Modified Serre–Green–Naghdi equations with improved or without dispersion

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Part I. Dispersion-improved model:

Improved Serre–Green–Naghdi equations.

Part II. Dispersionless model:

Regularised Saint-Venant–Airy equations.

Motivation

Understanding water waves (in shallow water).

Analytical approximations:

- Qualitative description;
- Physical insights.

Simplified equations:

- Easier numerical resolution;
- Faster schemes.

Goal:

- Derivation of the most accurate simplest models.

Hypothesis

Physical assumptions:

- Fluid is ideal, homogeneous & incompressible;
- Flow is irrotational, i.e., $\vec{V} = \operatorname{grad}\phi$;
- Free surface is a graph;
- Atmospheric pressure is constant.

Surface tension could also be included.

Notations for 2D surface waves over a flat bottom

- x : Horizontal coordinate.
- y : Upward vertical coordinate.
- t : Time.
- u : Horizontal velocity.
- v : Vertical velocity.
- ϕ : Velocity potential.
- $y = \eta(x,t)$: Equation of the free surface.
- $y = -d$: Equation of the seabed.
- Over tildes : Quantities at the surface, e.g., $\tilde{u} = u(y = \eta)$.
- Over check : Quantities at the surface, e.g., $\check{u} = u(y = -d)$.
- Over bar : Quantities averaged over the depth, e.g.,

$$\bar{u} \;=\; \frac{1}{h}\int_{-d}^{\eta} u \,\mathrm{d}y, \qquad h \;=\; \eta + d.$$

Mathematical formulation

- Continuity and irrotationality equations for $-d \leqslant y \leqslant \eta$

$$u_x = -v_y, \quad v_x = u_y \quad \Rightarrow \quad \phi_{xx} + \phi_{yy} = 0$$

- Bottom's impermeability condition at $y = -d$

$$\check{v} = 0$$

- Free surface's impermeability condition at $y = \eta(x,t)$

$$\eta_t + \tilde{u}\, \eta_x = \tilde{v}$$

- Dynamic free surface condition at $y = \eta(x,t)$

$$\phi_t + \tfrac{1}{2}\, \tilde{u}^2 + \tfrac{1}{2}\, \tilde{v}^2 + g\, \eta = 0$$

Shallow water scaling

Assumptions for large long waves in shallow water:

$$\sigma \;\propto\; \frac{\text{depth}}{\text{wavelength}} \ll 1 \quad \text{(shallowness parameter)},$$

$$\varepsilon \;\propto\; \frac{\text{amplitude}}{\text{depth}} = \mathcal{O}\left(\sigma^0\right) \quad \text{(steepness parameter)}.$$

Scale of derivatives and dependent variables:

$$\{\,\partial_x\,;\,\partial_t\,\} \;=\; \mathcal{O}\left(\sigma^1\right), \qquad \partial_y \;=\; \mathcal{O}\left(\sigma^0\right),$$

$$\{\,u\,;\,v\,;\,\eta\,\} \;=\; \mathcal{O}\left(\sigma^0\right), \qquad \phi \;=\; \mathcal{O}\left(\sigma^{-1}\right).$$

Solution of the Laplace equation and bottom impermeability

Taylor expansion around the bottom (Lagrange 1791):

$$u \;=\; \cos[\,(y+d)\,\partial_x\,]\,\check{u} \;=\; \check{u} \;-\; \tfrac{1}{2}\,(y+d)^2\,\check{u}_{xx} \;+\; \tfrac{1}{6}\,(y+d)^4\,\check{u}_{xxxx} \;+\; \cdots\,.$$

Low-order approximations for long waves:

$$u \;=\; \bar{u} \;+\; \mathcal{O}\big(\sigma^2\big)\,, \qquad \text{(horizontal velocity)}$$

$$v \;=\; -(y+d)\,\bar{u}_x \;+\; \mathcal{O}\big(\sigma^3\big)\,, \qquad \text{(vertical velocity).}$$

Energies

Kinetic energy:

$$\mathscr{K} \;=\; \int_{-d}^{\eta} \frac{u^2+v^2}{2}\,\mathrm{d}y \;=\; \frac{h\,\bar{u}^2}{2} \;+\; \frac{h^3\,\bar{u}_x^{\,2}}{6} \;+\; \mathcal{O}\!\left(\sigma^4\right),$$

Potential energy:

$$\mathscr{V} \;=\; \int_{-d}^{\eta} g\,(y+d)\,\mathrm{d}y \;=\; \frac{g\,h^2}{2}.$$

Lagrangian density (Hamilton principle):

$$\mathscr{L} \;=\; \mathscr{K} \;-\; \mathscr{V} \;+\; \{\,h_t \;+\; [h\bar{u}]_x\,\}\,\phi$$

Approximate Lagrangian

$$\mathscr{L}_2 \;=\; \tfrac{1}{2}h\bar{u}^2 \;-\; \tfrac{1}{2}gh^2 \;+\; \{h_t \;+\; [h\bar{u}]_x\}\,\phi \;+\; \mathcal{O}(\sigma^2).$$

$\Rightarrow$ Saint-Venant (non-dispersive) equations.

$$\mathscr{L}_4 \;=\; \mathscr{L}_2 \;+\; \tfrac{1}{6}\,h^3\,\bar{u}_x^2 \;+\; \mathcal{O}(\sigma^4).$$

$\Rightarrow$ Serre (dispersive) equations.

$$\mathscr{L}_6 \;=\; \mathscr{L}_4 \;-\; \tfrac{1}{90}\,h^5\,\bar{u}_{xx}^2 \;+\; \mathcal{O}(\sigma^6).$$

$\Rightarrow$ Extended Serre (ill-posed) equations.

Serre equations derived from $\mathscr{L}_4$

Euler–Lagrange equations yield:

$$
\begin{aligned}
0 \;&=\; h_t \;+\; \partial_x[\,h\,\bar{u}\,]\,, \\
0 \;&=\; \partial_t\big[\,\bar{u} \;-\; \tfrac{1}{3}\,h^{-1}(h^3\bar{u}_x)_x\,\big] \\
&\qquad +\; \partial_x\big[\,\tfrac{1}{2}\,\bar{u}^2 \;+\; g\,h \;-\; \tfrac{1}{2}\,h^2\,\bar{u}_x^2 \;-\; \tfrac{1}{3}\,\bar{u}\,h^{-1}(h^3\bar{u}_x)_x\,\big]\,.
\end{aligned}
$$

Secondary equations:

$$
\begin{aligned}
\bar{u}_t \;+\; \bar{u}\,\bar{u}_x \;+\; g\,h_x \;+\; \tfrac{1}{3}\,h^{-1}\,\partial_x\big[\,h^2\,\gamma\,\big] \;&=\; 0, \\
\partial_t[\,h\,\bar{u}\,] \;+\; \partial_x\big[\,h\,\bar{u}^2 \;+\; \tfrac{1}{2}\,g\,h^2 \;+\; \tfrac{1}{3}\,h^2\,\gamma\,\big] \;&=\; 0, \\
\partial_t\big[\,\tfrac{1}{2}\,h\,\bar{u}^2 \;+\; \tfrac{1}{6}\,h^3\bar{u}_x^2 \;+\; \tfrac{1}{2}\,g\,h^2\,\big] \;+& \\
\partial_x\big[\,(\tfrac{1}{2}\,\bar{u}^2 \;+\; \tfrac{1}{6}\,h^2\,\bar{u}_x^2 + g\,h \;+\; \tfrac{1}{3}\,h\,\gamma\,)\,h\,\bar{u}\,\big] \;&=\; 0,
\end{aligned}
$$

with

$$
\gamma \;=\; h\,\big[\,\bar{u}_x^2 \;-\; \bar{u}_{xt} \;-\; \bar{u}\,\bar{u}_{xx}\,\big]\,.
$$

2D Serre's equations on flat bottom (summary)

Easy derivations via a variational principle.

Non-canonical Hamiltonian structure.

(Li, J. Nonlinear Math. Phys., 2002)

Multi-symplectic structure.

(Chhay, Dutykh & Clamond, J. Phys. A, 2016)

Fully nonlinear, weakly dispersive.

(Wu, Adv. App. Mech. 37, 2001)

Can the dispersion be improved?

Modified vertical acceleration

$$\gamma \;=\; 2\,h\,\bar{u}_x^2 \;-\; h\,\partial_x\,[\,\bar{u}_t \,+\, \bar{u}\,\bar{u}_x\,] \,+\, \mathcal{O}(\sigma^4).$$

Horizontal momentum:

$$\underbrace{\bar{u}_t \,+\, \bar{u}\,\bar{u}_x}_{\mathcal{O}(\sigma)} \;=\; \underbrace{-\,g\,h_x}_{\mathcal{O}(\sigma)} \;-\; \underbrace{\tfrac{1}{3}\,h^{-1}\,\partial_x\big[\,h^2\,\gamma\,\big]}_{\mathcal{O}(\sigma^3)} \,+\, \mathcal{O}(\sigma^5).$$

Alternative vertical acceleration at the free surface:

$$\gamma \;=\; 2\,h\,\bar{u}_x^2 \;+\; g\,h\,h_{xx} \;+\; \mathcal{O}(\sigma^4).$$

Generalised vertical acceleration at the free surface:

$$\gamma \;=\; 2\,h\,\bar{u}_x^2 \;+\; \beta\,g\,h\,h_{xx} \;+\; (\beta-1)\,h\,\partial_x[\,\bar{u}_t \,+\, \bar{u}\,\bar{u}_x\,] \;+\; \mathcal{O}(\sigma^4).$$

β: free parameter.

Modified Lagrangian

Substitute $h\bar{u}_x^2 = \gamma + h(\bar{u}_{xt} + \bar{u}\bar{u}_{xx})$:

$$\mathscr{L}_4 = \frac{h\,\bar{u}^2}{2} + \frac{h^2\,\gamma}{12} + \frac{h^3}{12}\,[\,\bar{u}_t + \bar{u}\,\bar{u}_x\,]_x - \frac{g\,h^2}{2} + \{\,h_t + [\,h\,\bar{u}\,]_x\,\}\,\phi.$$

Substitution of the generalised acceleration:

$$\gamma = 2\,h\,\bar{u}_x^2 + \beta\,g\,h\,h_{xx} + (\beta - 1)\,h\,\partial_x[\,\bar{u}_t + \bar{u}\,\bar{u}_x\,] + \mathcal{O}\big(\sigma^4\big).$$

Resulting Lagrangian:

$$\mathscr{L}_4' = \mathscr{L}_4 + \underbrace{\frac{\beta\,h^3}{12}\,[\,\bar{u}_t + \bar{u}\,\bar{u}_x + g\,h_x\,]_x}_{\mathcal{O}(\sigma^4)} + \mathcal{O}\big(\sigma^4\big).$$

Reduced modified Lagrangian

After integrations by parts and neglecting boundary terms:

$$\begin{aligned}\mathscr{L}_4'' \;&=\; \frac{h\,\bar{u}^2}{2} \;+\; \frac{(2+3\beta)\,h^3\,\bar{u}_x^{\,2}}{12} \;-\; \frac{g\,h^2}{2} \;-\; \frac{\beta\,g\,h^2\,h_x^{\,2}}{4} \\ &\quad +\; \{\,h_t \,+\, [\,h\,\bar{u}\,]_x\,\}\,\phi \\ &=\; \mathscr{L}_4' \;+\; \text{boundary terms.}\end{aligned}$$

Equations of motion

$$
\begin{aligned}
h_t \,+\, \partial_x[h\bar{u}] \,&=\, 0, \\
q_t \,+\, \partial_x\!\left[\bar{u}q - \tfrac{1}{2}\bar{u}^2 + gh - \left(\tfrac{1}{2} + \tfrac{3}{4}\beta\right) h^2\bar{u}_x^{\,2} - \tfrac{1}{2}\beta g(h^2h_{xx} + hh_x^{\,2})\right] \,&=\, 0, \\
\bar{u}_t \,+\, \bar{u}\bar{u}_x \,+\, gh_x \,+\, \tfrac{1}{3}h^{-1}\partial_x\!\left[h^2\Gamma\right] \,&=\, 0, \\
\partial_t[h\bar{u}] + \partial_x\!\left[h\bar{u}^2 + \tfrac{1}{2}gh^2 + \tfrac{1}{3}h^2\Gamma\right] \,&=\, 0, \\
\partial_t\!\left[\tfrac{1}{2}h\bar{u}^2 + (\tfrac{1}{6} + \tfrac{1}{4}\beta)h^3\bar{u}_x^{\,2} + \tfrac{1}{2}gh^2 + \tfrac{1}{4}\beta gh^2h_x^{\,2}\right] + \\
\partial_x\!\left[\left(\tfrac{1}{2}\bar{u}^2 + (\tfrac{1}{6} + \tfrac{1}{4}\beta)h^2\bar{u}_x^{\,2} + gh + \tfrac{1}{4}\beta ghh_x^{\,2} + \tfrac{1}{3}h\Gamma\right) h\bar{u} + \tfrac{1}{2}\beta gh^3h_x\bar{u}_x\right] \,&=\, 0,
\end{aligned}
$$

where

$$
\begin{aligned}
q \,&=\, \phi_x \,=\, \bar{u} \,-\, \left(\tfrac{1}{3} + \tfrac{1}{2}\beta\right) h^{-1} \left[h^3\bar{u}_x\right]_x, \\
\Gamma \,&=\, \left(1 + \tfrac{3}{2}\beta\right) h \left[\bar{u}_x^{\,2} - \bar{u}_{xt} - \bar{u}\bar{u}_{xx}\right] - \tfrac{3}{2}\beta g \left[hh_{xx} + \tfrac{1}{2}\,h_x^{\,2}\right].
\end{aligned}
$$

Linearised equations

With $h = d + \eta$, η and $\bar{u}$ small, the equations become

$$\begin{aligned} \eta_t \; + \; d\,\bar{u}_x \; &= \; 0, \\ \bar{u}_t \; - \; \left(\tfrac{1}{3} + \tfrac{1}{2}\beta\right) d^2\,\bar{u}_{xxt} \; + \; g\,\eta_x \; - \; \tfrac{1}{2}\,\beta\,g\,d^2\,\eta_{xxx} \; &= \; 0. \end{aligned}$$

Dispersion relation:

$$\frac{c^2}{g\,d} \; = \; \frac{2 \, + \, \beta\,(kd)^2}{2 \, + \, \left(\tfrac{2}{3} + \beta\right)(kd)^2} \; \approx \; 1 \; - \; \frac{(kd)^2}{3} \; + \; \left(\frac{1}{3} + \frac{\beta}{2}\right) \frac{(kd)^4}{3}.$$

Exact linear dispersion relation:

$$\frac{c^2}{g\,d} \; = \; \frac{\tanh(kd)}{kd} \; \approx \; 1 \; - \; \frac{(kd)^2}{3} \; + \; \frac{2\,(kd)^4}{15}.$$

$\beta = 2/15$ is the best choice.

Steady solitary waves

Equation:

$$\left(\frac{\mathrm{d}\,\eta}{\mathrm{d}x}\right)^2 \;=\; \frac{(\mathcal{F}-1)\,(\eta/d)^2 \;-\; (\eta/d)^3}{\left(\frac{1}{3}+\frac{1}{2}\beta\right)\mathcal{F} \;-\; \frac{1}{2}\,\beta\,(1+\eta/d)^3},$$

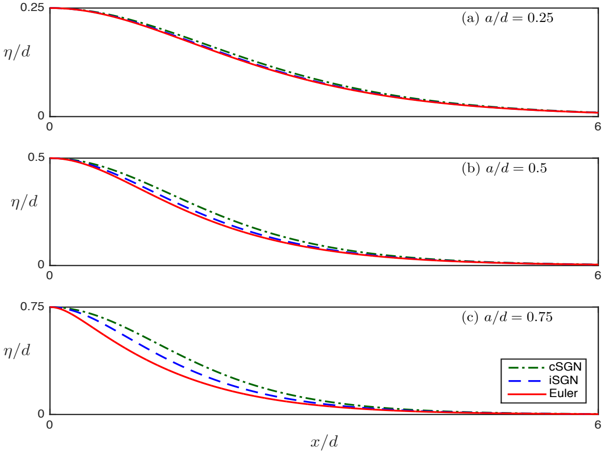
$$\mathcal{F} \;=\; c^2\,/\,g\,d.$$

Solution in parametric form:

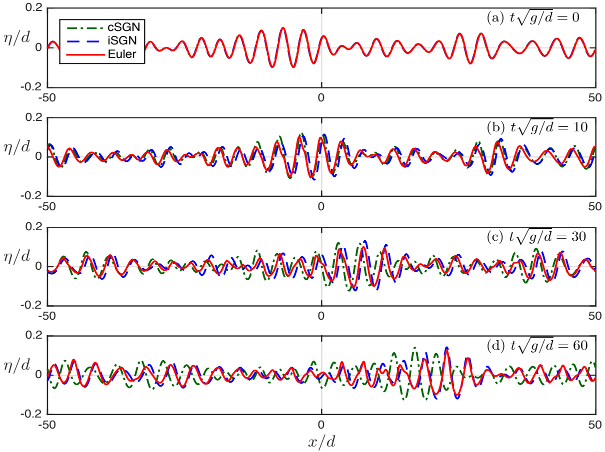
$$\frac{\eta(\xi)}{d} \;=\; (\mathcal{F}-1)\,\mathrm{sech}^2\!\left(\frac{\kappa\,\xi}{2}\right), \qquad (\kappa d)^2 \;=\; \frac{6\,(\mathcal{F}-1)}{(2+3\beta)\,\mathcal{F} \;-\; 3\,\beta}.$$

$$x(\xi) \;=\; \int_0^{\xi} \left|\, \frac{(\beta+2/3)\mathcal{F} \;-\; \beta\,h^3(\xi')/d^3}{(\beta+2/3)\,\mathcal{F} \;-\; \beta} \,\right|^{1/2} \mathrm{d}\xi',$$

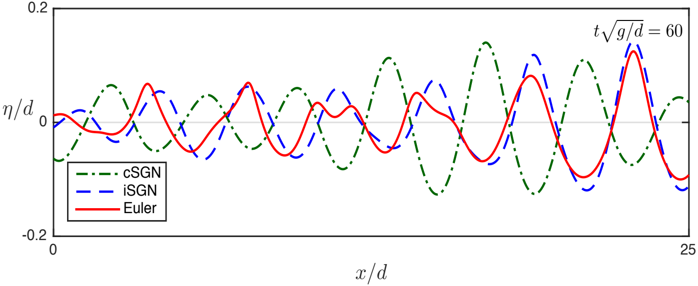
Comparisons for $\beta = 0$ and $\beta = 2/15$



Random wave field



Random wave field (zoom)



Part II

Saint-Venant equations

Non-dispersive shallow water (Saint-Venant) equations:

$$
\begin{aligned}
h_t \;+\; [\,h\,\bar{u}\,]_x \;&=\; 0,\\
\bar{u}_t \;+\; \bar{u}\,\bar{u}_x \;+\; g\,h_x \;&=\; 0.
\end{aligned}
$$

Shortcomings:

– No permanent regular solutions;
– Shocks appear;
– Requires ad hoc numerical schemes.

Regularisations

Add diffusion (e.g. von Neumann & Richtmayer 1950):

$$\bar{u}_t + \bar{u}\,\bar{u}_x + g\,h_x = \nu\,\bar{u}_{xx}.$$

$\Rightarrow$ Leads to dissipation of energy = bad for long time simulation.

Add dispersion (e.g. Lax & Levermore 1983):

$$\bar{u}_t + \bar{u}\,\bar{u}_x + g\,h_x = \tau\,\bar{u}_{xxx}.$$

$\Rightarrow$ Leads to spurious oscillations. Not always sufficient for regularisation.

Add diffusion + dispersion (e.g. Hayes & LeFloch 2000):

$\Rightarrow$ Regularises but does not conserve energy and provides spurious oscillations.

Leray-like regularisation

Bhat & Fetecau 2009 (J. Math. Anal. & App. 358):

$$
\begin{aligned}
h_t \,+\, [\,h\,\bar{u}\,]_x \,&=\, \epsilon^2\,h\,\bar{u}_{xxx}, \\
\bar{u}_t \,+\, \bar{u}\,\bar{u}_x \,+\, g\,h_x \,&=\, \epsilon^2\,(\,\bar{u}_{xxt} \,+\, \bar{u}\,\bar{u}_{xxx}\,)\,.
\end{aligned}
$$

Drawbacks:

– Shocks do not propagate at the right speed;
– No equation for energy conservation.

Dispersionless model

Two-parameter Lagrangian:

$$\mathscr{L} = \tfrac{1}{2}\,h\,\bar{u}^2 + \left(\tfrac{1}{6} + \tfrac{1}{4}\beta_1\right) h^3\,\bar{u}_x^{\,2} - \tfrac{1}{2}\,g\,h^2 \left(1 + \tfrac{1}{2}\beta_2\,h_x^{\,2}\right) + \left\{\, h_t + [\,h\,\bar{u}\,]_x \right\} \phi.$$

Linear dispersion relation:

$$\frac{c^2}{g\,d} = \frac{2 + \beta_2\,(kd)^2}{2 + \left(\tfrac{2}{3} + \beta_1\right)(kd)^2}.$$

No dispersion if $c = \sqrt{gd}$, that is $\beta_1 = \beta_2 - 2/3$, so let be

$$\beta_1 = 2\,\epsilon - 2/3, \qquad \beta_2 = 2\,\epsilon.$$

Conservative regularised SV equations

Regularised Saint-Venant equations:

$$
\begin{aligned}
0 \;&=\; h_t \;+\; \partial_x[\,h\,\bar{u}\,]\,, \\
0 \;&=\; \partial_t[\,h\,\bar{u}\,] \;+\; \partial_x\Big[\,h\,\bar{u}^2 \;+\; \tfrac{1}{2}\,g\,h^2 \;+\; \epsilon\,R\,h^2\,\Big]\,, \\
R \;&\stackrel{\text{def}}{=}\; h\left(\,\bar{u}_x^{\,2} \;-\; \bar{u}_{xt} \;-\; \bar{u}\,\bar{u}_{xx}\,\right) \;-\; g\left(\,h\,h_{xx} \;+\; \tfrac{1}{2}\,h_x^{\,2}\,\right).
\end{aligned}
$$

Energy equation:

$$
\begin{aligned}
&\partial_t\Big[\,\tfrac{1}{2}\,h\,\bar{u}^2 \;+\; \tfrac{1}{2}\,g\,h^2 \;+\; \tfrac{1}{2}\,\epsilon\,h^3\,\bar{u}_x^{\,2} \;+\; \tfrac{1}{2}\,\epsilon\,g\,h^2\,h_x^{\,2}\,\Big] \\
&+\,\partial_x\Big[\Big\{\,\tfrac{1}{2}\,\bar{u}^2 \;+\; g\,h \;+\; \tfrac{1}{2}\,\epsilon\,h^2\,\bar{u}_x^{\,2} \;+\; \tfrac{1}{2}\,\epsilon\,g\,h\,h_x^{\,2} \;+\; \epsilon\,h\,R\,\Big\}\,h\,\bar{u} \\
&\qquad +\,\epsilon\,g\,h^3\,h_x\,\bar{u}_x\,\Big] \;=\; 0.
\end{aligned}
$$

Permanent solutions

Non-dispersive solitary wave:

$$
\begin{aligned}
x \;&=\; \int_0^{\xi} \left[\, \frac{\mathcal{F} d^3 \;-\; h^3(\xi')}{(\mathcal{F}-1)\, d^3} \,\right]^{\frac{1}{2}} \mathrm{d}\xi', \\
\frac{\eta(\xi)}{d} \;&=\; (\mathcal{F}-1)\ \mathrm{sech}^2\!\left(\frac{\kappa\,\xi}{2}\right), \\
(\kappa d)^2 \;&=\; \epsilon^{-1}, \\
\mathcal{F} \;&=\; 1 \;+\; a\,/\,d.
\end{aligned}
$$

Example: Dam break problem

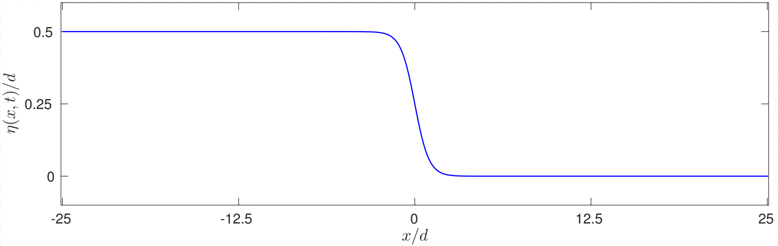
Initial condition:

$$h_0(x) = h_\mathrm{l} + \tfrac{1}{2}\,(h_\mathrm{r} - h_\mathrm{l})\,(1 + \tanh(\delta x))\,,$$
$$\bar{u}_0(x) = u_\mathrm{l} + \tfrac{1}{2}\,(u_\mathrm{r} - u_\mathrm{l})\,(1 + \tanh(\delta x))\,.$$

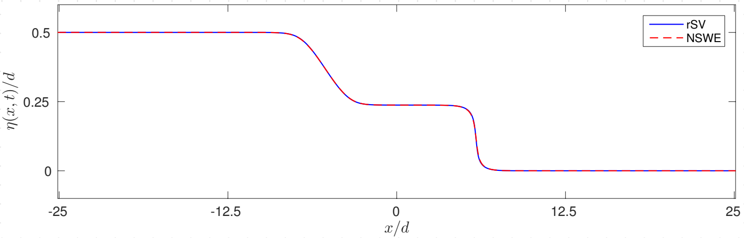
Resolution of the classical shallow water equations with finite volumes.

Resolution of the regularised equations with pseudo-spectral scheme.

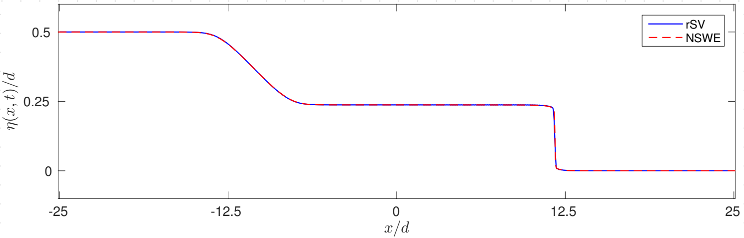
Result with $\epsilon = 0.001$ at $t\sqrt{g/d} = 0$



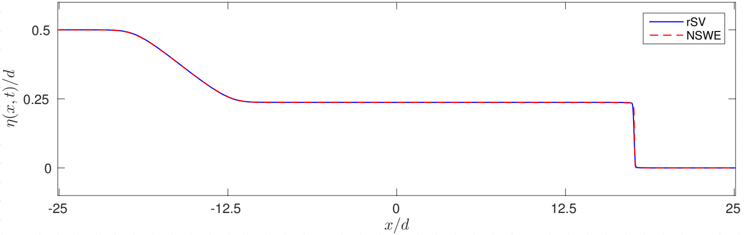
Result with $\epsilon = 0.001$ at $t\sqrt{g/d} = 5$



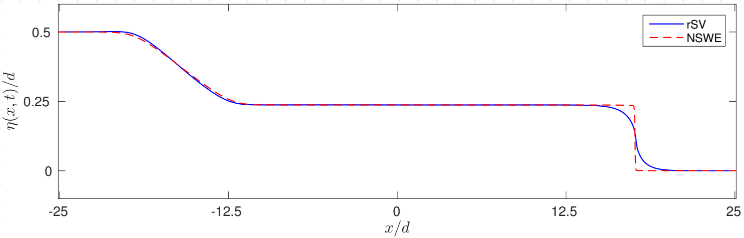
Result with $\epsilon = 0.001$ at $t\sqrt{g/d} = 10$



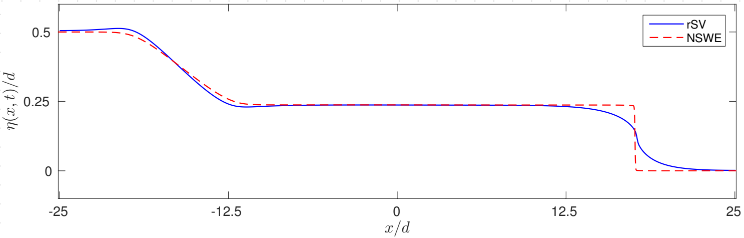
Result with $\epsilon = 0.001$ at $t\sqrt{g/d} = 15$



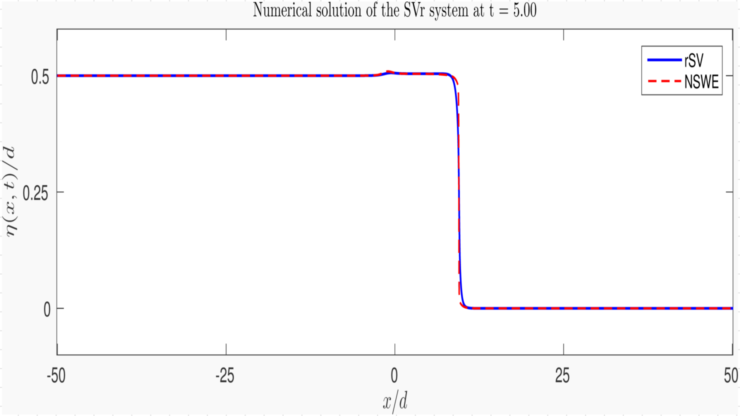
Result with $\epsilon = 1$ at $t\sqrt{g/d} = 15$



Result with $\epsilon = 5$ at $t\sqrt{g/d} = 15$



Example 2: Shock ($\epsilon = 0.1$)



Rankine–Hugoniot conditions

Assuming discontinuities in second (or higher) derivatives:

$$
\begin{aligned}
(u - \dot{s})\,[\![\, h_{xx} \,]\!] \;+\; h\,[\![\, u_{xx} \,]\!] \;&=\; 0, \\
(u - \dot{s})\,[\![\, u_{xx} \,]\!] \;+\; g\,[\![\, h_{xx} \,]\!] \;&=\; 0,
\end{aligned}
$$

$$
\Rightarrow \qquad \dot{s}(t) \;=\; u(x,t) \;\pm\; \sqrt{g\,h(x,t)} \qquad \text{at} \quad x \;=\; s(t).
$$

The regularised shock speed is **independent** of ϵ and the it propagates **exactly** along the **characteristic lines** of the Saint-Venant equations!

Summary

Variational principle yields:

— Easy derivations;
— Structure preservation;
— Suitable for enhancing models in a “robust way”.

Straightforward generalisations:

— 3D;
— Variable bottom;
— Stratification.

References

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CLAMOND, D. & DUTYKH, D. 2017. Non-dispersive conservative regularisation of nonlinear shallow water and isothermal Euler equations. *https://arxiv.org/abs/1704.05290.*